

Sectional Anosov flows: Existence of Venice masks with two singularities

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Abstract

We show the existence of venice masks (i.e. nontransitive sectional Anosov flows with dense periodic orbits, [3], [8], [9],[2]) containing two equilibria on certain compact 3-manifolds. Indeed, we present two type of examples in which the homoclinic classes composing their maximal invariant set intersect in a very different way.

1 Introduction

The dynamical systems theory describes different properties about asymptotic behavior, stability, relationships among system's elements and its properties. It is well known that the hyperbolic systems own some features and properties that provide very important information about its behavior. With the purpose of extending the notion of hyperbolicity, arise definitions and a new theory, such as partial hyperbolicity, singular hyperbolicity and sectional hyperbolicity. Thus, we begin by considering the relationship between the hyperbolic and sectional hyperbolic theory. Recall, the sectional hyperbolic sets and sectional Anosov flows were introduced in [7] and [6] respectively as a generalization of the hyperbolic sets and Anosov flows. They contain important examples such as the saddle-type hyperbolic attracting sets, the geometric and multidimensional Lorenz attractors [1], [4], [5]. A natural way is to observe the properties that are preserved or which are not in the new scenario. Particularly, we mention two important properties related to hyperbolic sets which are not satisfied by all sectional hyperbolic sets. The first is the spectral decomposition theorem [16]. It says that an attracting hyperbolic set $\Lambda = Cl(Per(X))$ is a finite disjoint union of homoclinic classes, where $Per(X)$ is the set of periodic points of X . The second says that an Anosov flow on a closed manifold is transitive if and only

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if it has dense periodic orbits. This results are false for sectional Anosov flows, i.e., sets whose maximal invariant is a sectional-hyperbolic set [9]. Specifically, it is proved that there exists a sectional Anosov flow such that it is supported on a compact 3-manifold, it has dense periodic orbits, is the union non disjoint of two homoclinic classes but is not transitive. So, a sectional Anosov flow is said a *Venice mask* if it has dense periodic orbits but is not transitive. The only known examples of venice masks have one or three singularities, and they are characterized by having two properties: are the union non disjoint of two homoclinic classes and the intersection of its homoclinic classes is the closure of the unstable manifold of a singularity [8], [9], [3]. Thus, we provide two examples of venice masks with two singularities, but with different features. In particular, each one is the union of two different homoclinic classes. However, for the first, the intersection of homoclinic classes is the closure of the unstable manifold of two singularities. Whereas for the second, the intersection of homoclinic classes is just a hyperbolic periodic orbit.

Let us state our results in a more precise way.

Consider a Riemannian compact manifold M of dimension three (a *compact 3-manifold* for short). We denote by ∂M the boundary of M . Let $\mathcal{X}^1(M)$ be the space of C^1 vector fields in M endowed with the C^1 topology. Fix $X \in \mathcal{X}^1(M)$, inwardly transverse to the boundary ∂M and denotes by X_t the flow of X , $t \in \mathbb{R}$.

The ω -limit set of $p \in M$ is the set $\omega_X(p)$ formed by those $q \in M$ such that $q = \lim_{n \rightarrow \infty} X_{t_n}(p)$ for some sequence $t_n \rightarrow \infty$. The α -limit set of $p \in M$ is the set $\alpha_X(p)$ formed by those $q \in M$ such that $q = \lim_{n \rightarrow \infty} X_{t_n}(p)$ for some sequence $t_n \rightarrow -\infty$. The *non-wandering set* of X is the set $\Omega(X)$ of points $p \in M$ such that for every neighborhood U of p and every $T > 0$ there is $t > T$ such that $X_t(U) \cap U \neq \emptyset$. Given $\Lambda \in M$ compact, we say that Λ is *invariant* if $X_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$. We also say that Λ is *transitive* if $\Lambda = \omega_X(p)$ for some $p \in \Lambda$; *singular* if it contains a singularity and *attracting* if $\Lambda = \bigcap_{t>0} X_t(U)$ for some compact neighborhood U of it. This neighborhood is often called *isolating block*. It is well known that the isolating block U can be chosen to be positively invariant, i.e., $X_t(U) \subset U$ for all $t > 0$. An *attractor* is a transitive attracting set. An attractor is *nontrivial* if it is not a closed orbit.

The *maximal invariant set* of X is defined by $M(X) = \bigcap_{t \geq 0} X_t(M)$.

Definition 1.1. A compact invariant set Λ of X is hyperbolic if there are a continuous tangent bundle invariant decomposition $T_\Lambda M = E^s \oplus E^X \oplus E^u$ and positive constants C, λ such that

- E^X is the vector field's direction over Λ .

- E^s is contracting, i.e., $\|DX_t(x)|_{E_x^s}\| \leq Ce^{-\lambda t}$, for all $x \in \Lambda$ and $t > 0$.
- E^u is expanding, i.e., $\|DX_{-t}(x)|_{E_x^u}\| \leq Ce^{-\lambda t}$, for all $x \in \Lambda$ and $t > 0$.

We denote by $m(L)$ the minimum co-norm of a linear operator L , i.e., $m(L) = \inf_{v \neq 0} \frac{\|Lv\|}{\|v\|}$.

Definition 1.2. A compact invariant set Λ of X is partially hyperbolic if there is a continuous invariant splitting $T_\Lambda M = E^s \oplus E^c$ with $E_x^c \neq 0$, $E_x^s \neq 0$ for all $x \in \Lambda$, such that the following properties hold for some positive constants C, λ :

- E^s is contracting, i.e., $\|DX_t(x)|_{E_x^s}\| \leq Ce^{-\lambda t}$, for all $x \in \Lambda$ and $t > 0$.
- E^s dominates E^c , i.e., $\frac{\|DX_t(x)|_{E_x^s}\|}{m(DX_t(x)|_{E_x^c})} \leq Ce^{-\lambda t}$, for all $x \in \Lambda$ and $t > 0$.

We say the central subbundle E_x^c of Λ is *sectionally-expanding* if

$$\dim(E_x^c) = 2$$

and

$$|\det(DX_t(x)|_{E_x^c})| \geq C^{-1}e^{\lambda t}, \quad \forall x \in \Lambda \quad \text{and} \quad t > 0.$$

Here $\det(DX_t(x)|_{E_x^c})$ denotes the jacobian of $DX_t(x)$ along E_x^c .

Definition 1.3. A sectional-hyperbolic set is a partially hyperbolic set whose singularities (if any) are hyperbolic and whose central subbundle is sectionally-expanding.

Definition 1.4. We say that X is a Anosov flow if $M(X) = M$ is a hyperbolic set. X is a sectional-Anosov flow if $M(X)$ is a sectional-hyperbolic set.

The Invariant Manifold Theorem [3] asserts that if x belongs to a hyperbolic set H of X , then the sets

$$W_X^{ss}(p) = \{x \in M : d(X_t(x), X_t(p)) \rightarrow 0, t \rightarrow \infty\} \quad \text{and}$$

$$W_X^{uu}(p) = \{x \in M : d(X_t(x), X_t(p)) \rightarrow 0, t \rightarrow -\infty\},$$

are C^1 immersed submanifolds of M which are tangent at p to the subspaces E_p^s and E_p^u of $T_p M$ respectively.

$$W_X^s(p) = \bigcup_{t \in \mathbb{R}} W_X^{ss}(X_t(p)) \quad W_X^u(p) = \bigcup_{t \in \mathbb{R}} W_X^{uu}(X_t(p))$$

are also C^1 immersed submanifolds tangent to $E_p^s \oplus E_p^X$ and $E_p^X \oplus E_p^u$ at p respectively.

Recall that a singularity of a vector field is hyperbolic if the eigenvalues of its linear part have non zero real part.

Definition 1.5. We say that a singularity σ of a sectional-Anosov flow X is Lorenz-like if it has three real eigenvalues $\lambda^{ss}, \lambda^s, \lambda^u$ with $\lambda^{ss} < \lambda^s < 0 < -\lambda^s < \lambda^u$. $W^s(\sigma)$ is the manifold associated to eigenvalues λ^{ss}, λ^s , and $W^{ss}(\sigma)$ is the manifold associated to eigenvalue λ^{ss} .

Definition 1.6. A periodic orbit of X is the orbit of some p for which there is a minimal $t > 0$ (called the period) such that $X_t(p) = p$.

A homoclinic orbit of a hyperbolic periodic orbit O is an orbit $\gamma \subset W^s(O) \cap W^u(O)$. If additionally $T_q M = T_q W^s(O) + T_q W^u(O)$ for some (and hence all) point $q \in \gamma$, then we say that γ is a transverse homoclinic orbit of O . The homoclinic class $H(O)$ of a hyperbolic periodic orbit O is the closure of the union of the transverse homoclinic orbits of O . We say that a set Λ is a homoclinic class if $\Lambda = H(O)$ for some hyperbolic periodic orbit O .

Definition 1.7. A Venice mask is a sectional-Anosov flow with dense periodic orbits which is not transitive.

With these definitions we can state our main results.

Theorem A. *There exists a Venice mask X with two singularities supported on a 3-manifold M , such that:*

- $M(X)$ is the union of two homoclinic classes $\mathcal{H}_X^1, \mathcal{H}_X^2$.
- $\mathcal{H}_X^1 \cap \mathcal{H}_X^2 = O$, where O is a hyperbolic periodic orbit.

Theorem B. *There exists a Venice mask Y with two singularities supported on a 3-manifold N , such that:*

- $N(Y)$ is the union of two homoclinic classes $\mathcal{H}_Y^1, \mathcal{H}_Y^2$.
- $\mathcal{H}_Y^1 \cap \mathcal{H}_Y^2 = Cl(W^u(\sigma_1) \cup W^u(\sigma_2))$, where σ_1, σ_2 are the singularities of Y .

In section 3.2, we shall be described briefly this construction by using one-dimensional and two-dimensional maps. In section 4.1, from modifications on the previous maps in Section 3.2 and by considering a plug, we shall prove the Theorem A. In the same way, in Section 4.2, by using the venice mask with a unique singularity, the Theorem B will be obtained by gluing a particular plug preserving the original flow.

2 Preliminaries

2.1 Original plugs

In order to obtain the three-dimensional vector field of our example, we begin by considering the well known *Plykin attractor* and the *Cherry flow* (See [14], [13]).

We give a sketch of the flow construction. It will be constructed through three steps, firstly by modifying the Cherry flow. In fact, we consider a vector field in the square whose flow is described in Figure 1 a). Note that this vector field has two equilibria: a saddle σ and a sink p . For σ one has that its eigenvalues $\{\lambda_s, \lambda_u\}$ of σ satisfy the relation

$$\lambda_s < 0 < -\lambda_s < \lambda_u.$$

We have depicted a small disk D centered at the attracting equilibrium p (Figure 1 b). Note that the flow is pointing inward the edge of the disk. This finishes the first step for the construction.

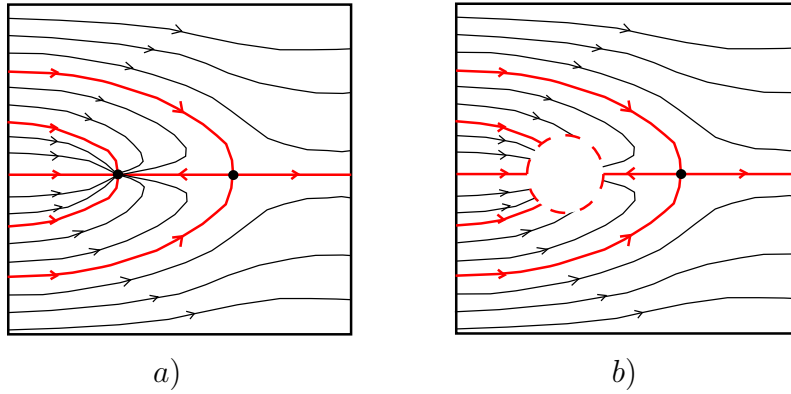


Figure 1: Cherry flow.

For the second step we multiply the above vector field by a strong contraction λ_{ss} in order to obtain the vector field described in Figure 2 a). We can choose λ_{ss} such that $-\lambda_{ss}$ be large, so the resulting vector field will have a Lorenz-like singularity and this new eigenvalue will be associated with the strong manifold of the singularity. This yields a Cherry flow box and finishes the second step for the construction.

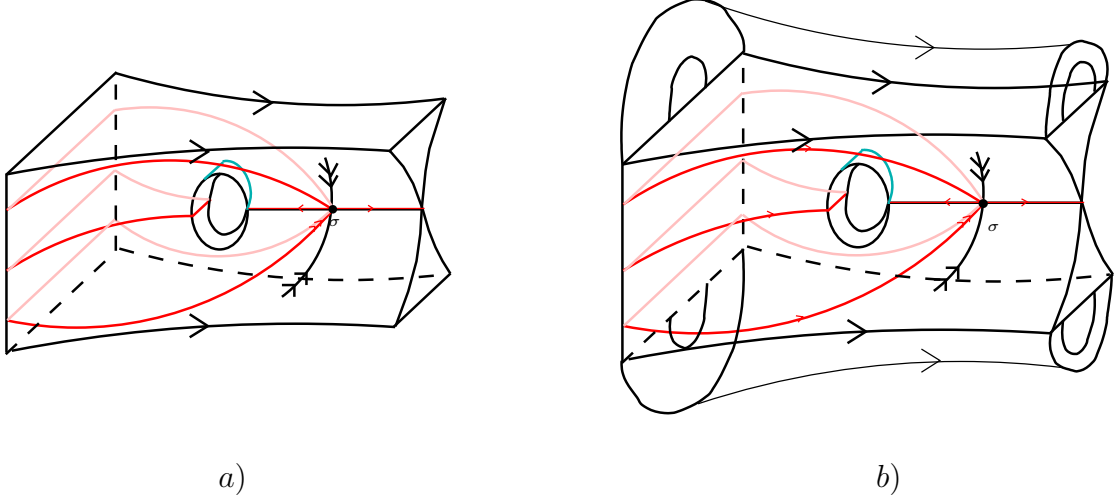


Figure 2: Cherry flow box and Plug 2.

From Plykin attractor follows that the construction must have at least two holes inasmuch as we will use certain return map. Then, the final step is to glue two handles that provides two holes and the three dimensional vector field above in order to obtain the vector field whose flow is given in Figure 2 b). Hereafter the resulting vector field will be called of Plug 2.

The hole indicated in this Figure 2 is nothing but the disk D times a compact interval I_1 . Again, note that the flow is pointing inward the edge of the hole by construction. For this reason, we take a solid 3-ball and we define a flow on this one. Indeed, it flow has no singularities, it acts as in Figure 3 and will be used for to glue the hole's bound with this one. Hereafter the resulting vector field will be called of Plug 3.

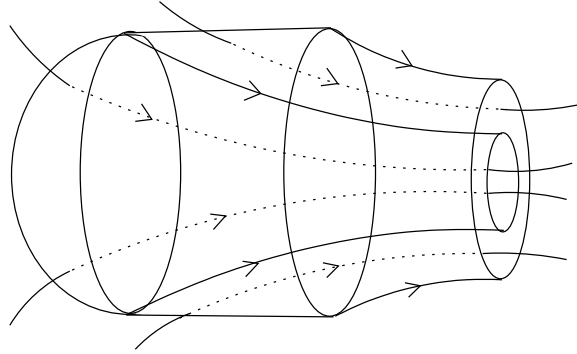


Figure 3: Plug 3.

3 Modified maps

We begin by considering the construction made in [3] like model in order of obtain the vector fields X and Y of the main theorems. Recall that the original model provides tools for a three dimensional example with a unique singularity. The aim main is modify the original maps, in order to make a suspension of the modify maps via the new plugs. For this purpose, we will do such modifications followed by its original maps.

3.1 One-dimensional map

Thus, in the same way of [3], we consider the branched 1-manifold \mathcal{B} consisting of a compact interval and a circle with branch point b . We cut \mathcal{B} open along b to obtain a compact interval which we assume to be $[0, 1]$ for simplicity. In $[0, 1]$ we consider three points $0 < d_1 < d_* < d_2 < 1$, where d_* is depicted also in the Figure 4. These will be the discontinuity points of f as a map of $[0, 1]$. The set $\mathcal{B} \setminus \{d_*\}$ will be the domain of f . We define $f : \mathcal{B} \setminus \{d_*\} \rightarrow \mathcal{B}$ in a way that its graph in $[0, 1]$ is the one in Figure 4.

By construction one has that f satisfies the following hypotheses:

(H1): $Dom(f) = [0, 1] \setminus \{d_*\}$.

(H2): $f(0) = 0$; $f(d_1) = f(d_2) = 1$; $f(1) = f(b) \in (0, d_1)$.

(H3): $f(d_1+) = f(d_2+) = b$; $f(d_1-) = f(d_2-) = 1$; $f(d_*+) = f(d_*-) = 0$.

(H4): $f([0, d_1]) = [0, 1]$; $f((d_1, d_*)) = (0, b)$; $f((d_*, d_2]) = (0, 1]$; $f((d_2, 1]) = [f(b), b)$.

(H5): f is expanding, i.e., f is C^1 in $Dom(f)$ and there is $\lambda > 1$ such that $|f'(x)| \geq \lambda$, for each $x \in Dom(f)$.

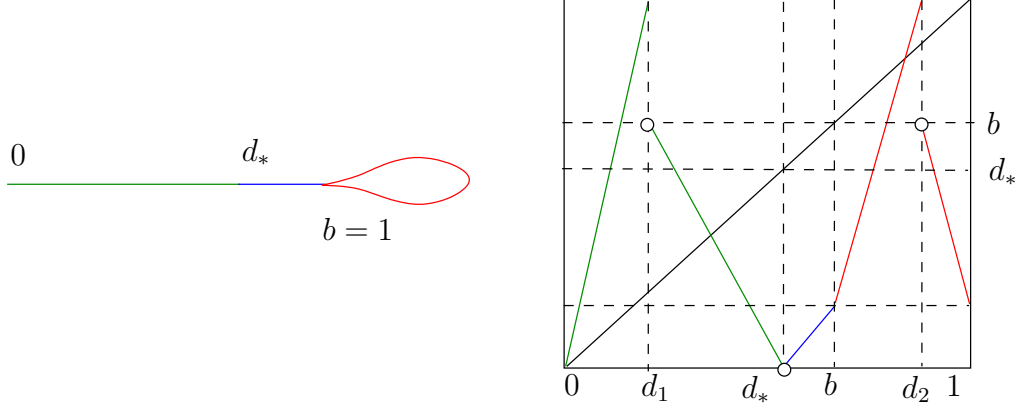


Figure 4: The quotient space and one-dimensional map.

3.2 Modified one-dimensional map

We realize a modification of the above map f . Denote $d_* = d^+$ and let $f^+ : \mathcal{B}^+ \setminus \{d^+\} \rightarrow \mathcal{B}^+$ be in a way that its graph in $[0, 1]$ is the one in Figure 5.

Here, there exist $\varepsilon > 0$ small such that $\int_0^{d_1} \sqrt{[(f)'(x)]^2 + 1} dx < \int_0^{d_1} \sqrt{[(f^+)'(x)]^2 + 1} dx < \int_0^{d_1} \sqrt{[(f)'(x)]^2 + 1} dx + \varepsilon$ and $\int_{d_*}^b \sqrt{[(f)'(x)]^2 + 1} dx < \int_{d_*}^b \sqrt{[(f^+)'(x)]^2 + 1} dx < \int_{d_*}^b \sqrt{[(f)'(x)]^2 + 1} dx + \varepsilon$. Moreover f^+ satisfies **(H1)**-**(H5)**. We define $f^-(x) = f^+(-x)$ and denote $-d^+ = d^-$. $f^- : \mathcal{B}^- \setminus \{d^-\} \rightarrow \mathcal{B}^-$.

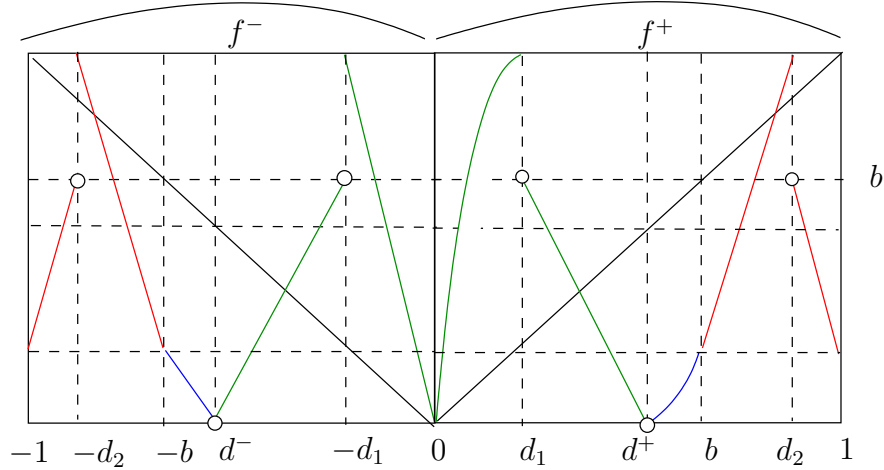


Figure 5: Modified one-dimensional map.

These following results examining the properties of f and appears in [3]. This in turns through a structure closely related to [3] and by construction we obtain the same properties for the f^+ map.

Definition 3.1. *We say that f is locally eventually onto (leo for short) if given any open interval $I \subset [0, 1]$ there is $m \geq 0$ such that $f^m(I) = [0, 1]$.*

Theorem 3.2. f^+ is leo.

Corollary 3.3. *The periodic points of f^+ are dense in \mathcal{B} . If $x \in \mathcal{B}$, then*

$$\mathcal{B} = Cl \left(\bigcup_{n \geq 0} (f^+)^{-n}(x) \right).$$

3.3 Two-dimensional map

We consider the twice punctured planar region R depicted in Figure 6. It is formed by: two half-annuli A, F , and four rectangles B, C, D, E . There is a middle vertical line denoted by l . Note that l defines a plane reflexion throughout denoted by θ . We assume $\theta(D) = C$, $\theta(E) = B$ and $\theta(F) = A$. In particular, $\theta(R) = R$ and $\theta(d^+) = d^-$, where the vertical segments d^-, d^+ correspond to the right-hand and left-hand boundary curves of B and D respectively. We define $H^- = A \cup B \cup C$ and $H^+ = D \cup E \cup F$.

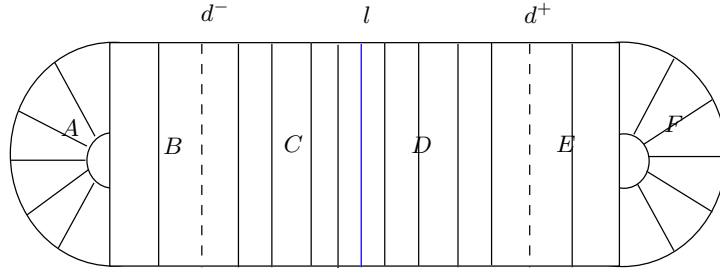


Figure 6: Region R .

In [3] was defined the C^∞ map $G : R \setminus \{d^-, d^+\} \rightarrow \text{Int}(R)$. It satisfies the following hypotheses:

- (G1): G and θ commute, i.e., $G \circ \theta = \theta \circ G$.
- (G2): G preserves and contracts the foliation \mathcal{F} .
- (G3): Let $g : K \setminus \{d^-, d^+\} \rightarrow K$ be the map induced by G in the leaf space K .

Then, the map f^+ defined by $f^+ = g|_{\mathcal{B}^+}$ satisfies the hypotheses **(H1)**-**(H5)**, with $f = f^+$, $\mathcal{B} = \mathcal{B}^+$ and $d_* = d^+$.

Properties of G

- By **(G1)**, H^+ and H^- are invariant under G .
- Since G contracts \mathcal{F} (**(G2)**) we have that $W^s(x, G)$ is union of leaves of \mathcal{F} . It follows from **(G2)**, **(G3)** and the expansiveness in **(H5)** that all periodic points of G are hyperbolic saddles.
- By **(G1)** we have that $G(l) \subset l$ and so G has a fixed point P in l . Clearly one has $\pi(P) = 0$.

Define

$$A_G^- = Cl \left(\bigcap_{n \geq 1} G^n(H^-) \right), \quad A_G^+ = Cl \left(\bigcap_{n \geq 1} G^n(H^+) \right).$$

Theorem 3.4. A_G^- and A_G^+ are homoclinic classes and $P \in A_G^+ \cap A_G^-$.

3.4 Modified two-dimensional map

For the region R in Figure 6, we define the C^∞ map $H : R \setminus \{d^-, d^+\} \rightarrow \text{Int}(R)$ in a way that its image is as indicated in Figure 7. We require the following hypotheses:

(L1): H^- , H^+ are invariant under H . $H(H^- \setminus \{d^-\}) \subset H^-$ and $H(H^+ \setminus \{d^+\}) \subset H^+$.

(L2): H preserves and contracts the foliation \mathcal{F} .

(L3): Let $h : K \setminus \{d^-, d^+\} \rightarrow K$ be the map induced by H in the leaf space K .

Then, the map $f^{+(-)}$ defined by $f^{+(-)} = h|_{\mathcal{B}^{+(-)}}$ satisfies the hypotheses **(H1)**-**(H5)**, $\mathcal{B} = \mathcal{B}^{+(-)}$ and $d_* = d^{+(-)}$.

We observe that **(L1)** implies $H(l) \subset l$ and by contraction, H has a fixed point $P \in l$. Again, for

$$A_H^- = Cl \left(\bigcap_{n \geq 1} H^n(H^-) \right), \quad A_H^+ = Cl \left(\bigcap_{n \geq 1} H^n(H^+) \right)$$

we have that A_H^+ and A_H^- are homoclinic classes and $\{P\} = A_H^+ \cap A_H^-$.

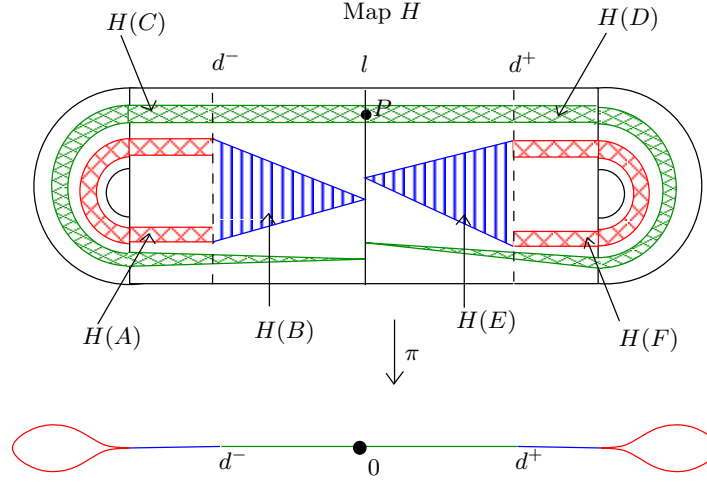


Figure 7: The quotient space and modified two-dimensional map.

3.5 Venice mask with one singularity

Recall, by considering the original maps (Subsection 3.1, 3.3), and by using the plugs 2, 3, in [3] was construct the venice mask example with one singularity. Here, we provides a graphic idea in order to compare it with the new examples.

The Figure 8 a) shows the flow, whereas the Figure 8 b) shows the ambient manifold that supports this one. The ambient manifold is a solid bi-torus excluding two tori neighborhoods V_1, V_2 associated to two repelling periodic orbits O_1, O_2 respectively.

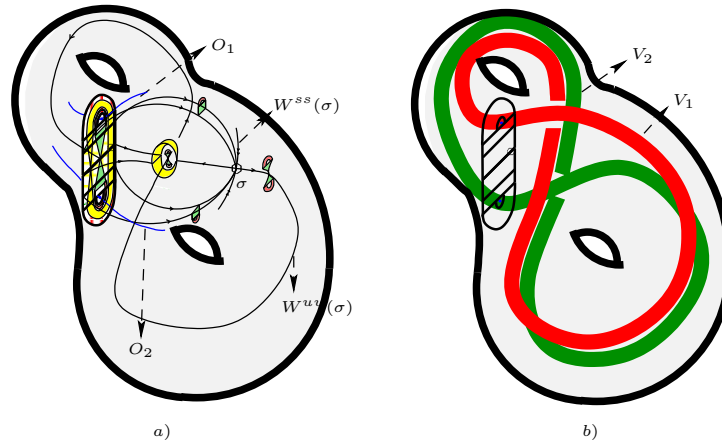


Figure 8: Venice mask with one singularity

4 Venice mask's examples with two singularities

4.1 Vector field X and Example 1.

In this section, we construct a vector field X which will satisfy the properties in the Theorem A by using the subsection 3.2 and 3.4.

We begin by considering a vector field as the Cherry flow described in Figure 1, with the same conditions of subsection 3.2.

We called this flow of A and we proceed to perturb it, following the ideas of the well known DA-Attractor introduced by Smale (see [14]). Let U be a neighborhood (relatively small) of σ . We can obtain a flow φ^t such that $\text{supp}(\varphi^t - id) \subset U$ (Figure 9 a)). Also, the derivate of the flow at σ with respect to canonical basis in $T_\sigma Q$ is

$$D\varphi_\sigma^t = \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix}.$$

We deform such a flow in order to obtain a one-parameter family of flows $B^t = \varphi^t \circ A$. Let $\tau > 0$ such that $e^\tau \lambda_s > 1$, so σ is a source for B^τ . Moreover, the new map has three fixed points on $W_X^s(\sigma)$, σ a source and σ_1, σ_2 saddles. Moreover, there exists a neighborhood V of σ (not containing σ_1 and σ_2) contained in U such that $B_s^\tau(V) \supset V$ for all $s > 0$ (Figure 9 b)). Thus, we obtain a vector field as the square Q whose flow A is described in Figure 9.

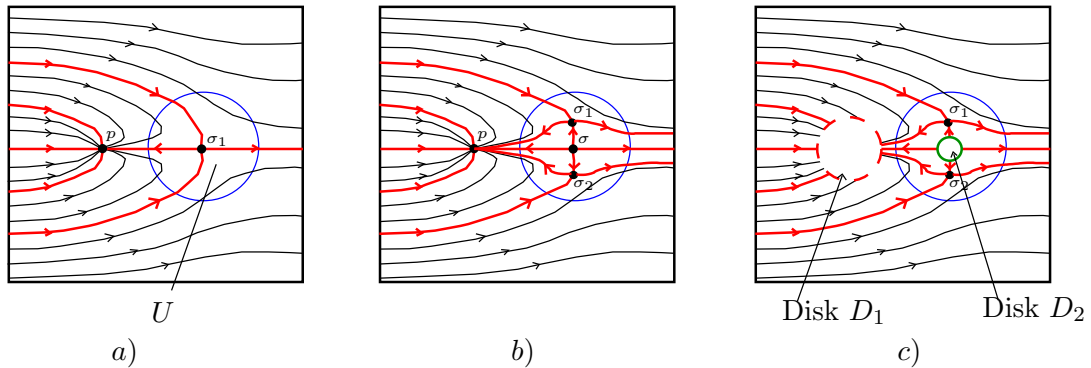


Figure 9: Perturbed Cherry flow

Now, we remove two small disks $D_1, D_2 = V$ centered at the attracting equilibrium p and at the repelling equilibrium σ respectively (Figure 9 c)).

In the next step, we multiply the above vector field by a strong contraction λ_{ss} in order to obtain the similar vector field described in Figure 2 b). We choose λ_{ss} such that σ_1 and σ_2 are Lorenz-like.

Now, we consider an interval $I_0 = I_1 \times \{p_0\}$, where p_0 is the point of intersection between $W_X^u(\sigma)$ and the disk D_1 . We realize a modification in the flow such that a branched of $W_X^u(\sigma_1)$ intersects a connected component of $I_0 \setminus \{p_0\}$ and a branched of $W_X^u(\sigma_2)$ intersects the other connected component of $I_0 \setminus \{p_0\}$ (See 10).

The final step is to glue two handles on the 3-dimensional vector field above in order to obtain the vector field whose flow is given in Figure 10 a). The resulting vector field is what we shall call Plug X.

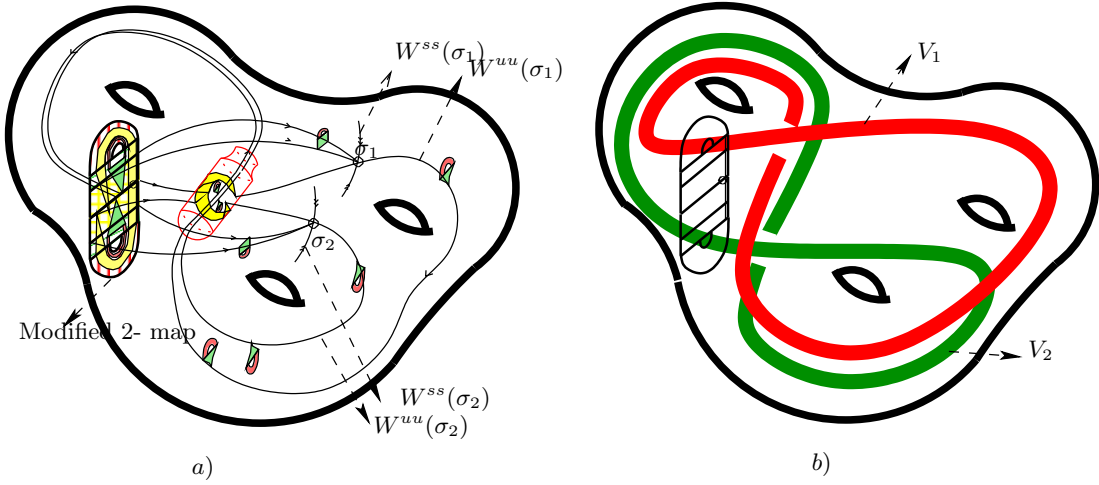


Figure 10: Plug X and its associated manifold.

In the same way as in Figure 2, in this case, by multiplying the above vector field by a strong contraction generate two holes and it is nothing but the disks D_1 times a compact interval I_1 , and D_2 times a compact interval I_2 . Also, let us to use the Plug 3 and apply on the hole associated to D_1 . Note that the interval I_2 is chosen such that $D_2 \times I_2$ produces the third hole on the ambient manifold. It generates a solid tritorus (see Figure 10 b)).

Then, we construct a vector field X on a solid tritorus ST_1 in a way that $X_t(ST_1) \subset \text{Int}(ST_1)$ for all $t > 0$ and X is transverse to the boundary of the solid tritorus. The flow is obtained gluing plugs X and 3 as indicated in Figure 10 a).

We require the following hypotheses:

(X1): There are two repelling periodic orbits O_1, O_2 in $\text{Int}(ST_1)$ crossing the holes of R .

(X2): There are two solid tori neighborhoods $V_1, V_2 \subset \text{Int}(ST_1)$ of O_1, O_2 with boundaries transverse to X_t such that if $M = ST_1 \setminus (V_1 \cup V_2)$, then M is a compact neighborhood with smooth boundary transverse to X_t and $X_t(M) \subset M$ for $t > 0$. As M is a solid torus with two solid tori removed, we have that M is connected as indicated in Figure 10 b).

(X3): $R \subset M$ and the return map H induced by X in R satisfies the properties **(L1)-(L3)** in Section 3.4. Moreover,

$$\{q \in M : X_t(q) \notin R, \forall t \in \mathbb{R}\} = \{\sigma_1, \sigma_2\}.$$

Now, define

$$A^+ = Cl\left(\bigcup_{t \in \mathbb{R}} X_t(A_H^+)\right) \quad \text{and} \quad A^- = Cl\left(\bigcup_{t \in \mathbb{R}} X_t(A_H^-)\right).$$

Proposition 4.1. $W_X^u(\sigma_1) \subset A^+$ and $W_X^u(\sigma_2) \subset A^-$.

Proof. If $x \in H^+$ is a periodic point of H , then $G^n(x) \in R$ for all $n \leq 0$ and so $x \in A_H^+ = Cl(\bigcap_{n \geq 1} H^n(H^+))$. Therefore $x \in A^+$ (for $A_{H^+} \subset A^+$) and by invariance of A^+ , the full orbit of x is contained in A^+ .

Second, the periodic points of f^+ in **(L3)** are dense in \mathcal{B} by Corollary 3.3. Then, the periodic points of H accumulate on d^+ in both connected components of $H^+ \setminus d^+$. Since d^+ is contained in $W_X^s(\sigma_1)$, the full X_t -orbit of the periodic points of H accumulating d^+ also accumulate on $W_X^u(\sigma_1)$. Then $W_X^u(\sigma_1) \subset A^+$ because A^+ is closed. Analogously, we have $W_X^u(\sigma_2) \subset A^-$. \square

Define $A_H = A_H^+ \cup A_H^-$ and

$$A = Cl\left(\bigcup_{t \in \mathbb{R}} X_t(A_H)\right),$$

Lemma 4.2. A^+ and A^- are homoclinic classes of X and $A = A^+ \cup A^-$.

Proof. See [3]. \square

Proposition 4.3. X is a sectional Anosov flow.

Proof. In the same way of [3], we will prove that A is a sectional-hyperbolic set and $M(X) = A$. Indeed, how $A = A_1 \cup A_2$ is union of homoclinic classes then A has dense periodic orbits (Birkhoff-Smale Theorem). Moreover, of the hypotheses **(L2)** and **(L3)** follows that every periodic orbit of X contained in A has a hyperbolic splitting $T_O M = E_O^s \oplus E_O^X \oplus E_O^u$. Here, E_O^s is due to **(L2)**, E_O^u by **(L3)** and E_O^X is the one-dimensional subbundle over O induced by X . Let $Per(A)$ be the union of the periodic orbits of X contained in A . Define the splitting

$$T_{Per(A)} M = F_{Per(A)}^s \oplus F_{Per(A)}^c,$$

where $F_x^s = E_x^s$ and $F_x^c = E_x^X \oplus E_x^u$ for $x \in Per(A)$. As every periodic orbit in M of every vector field C^1 close to X is hyperbolic of saddle type, we can use the arguments in [11] to prove that the splitting $T_{Per(A)} M = F_{Per(A)}^s \oplus F_{Per(A)}^c$ over $Per(A)$ extends to a sectional-hyperbolic splitting $T_A M = F_A^s \oplus F_A^c$ over the whole $A = Cl(Per(A))$.

We conclude that X is a sectional Anosov flow on M . □

Proof of Theorem A.

By using the Lemma 4.2 and the Proposition 4.3 we have that X is a sectional Anosov flow and $M(X)$ is the union of two homoclinic classes H_X^1, H_X^2 , where $H_X^1 = A^+$ and $H_X^2 = A^-$. Since $\{P\} = A_H^+ \cap A_H^-$, it implies that $H_X^1 \cap H_X^2 = O$, with O the orbit associated to P . In particular X is a Venice mask, and by construction it has two singularities.

4.2 Vector field Y and Example 2.

In this section, we construct a vector field Y which will satisfy the properties in the Theorem B by using the results from [3].

Firstly, in order to obtain the vector field Y , we begin by considering the venice mask with one singularity. Unlike the previous section, in this case we will not perturb the flow. Moreover, we will change the flow by preserving the plugs 2, 3 and we will remove a connected component of the flow and its ambient manifold.

The main aim of remove a connected component will be glue a new plug with different features, properties and that provides other singularity. This process is done in simple steps. (see Figure 11). Indeed, the important steps are Figure 11 c), d) and since we want a plug by containing a singularity, we will see that the this one has a hole, which is produced by the singularity.

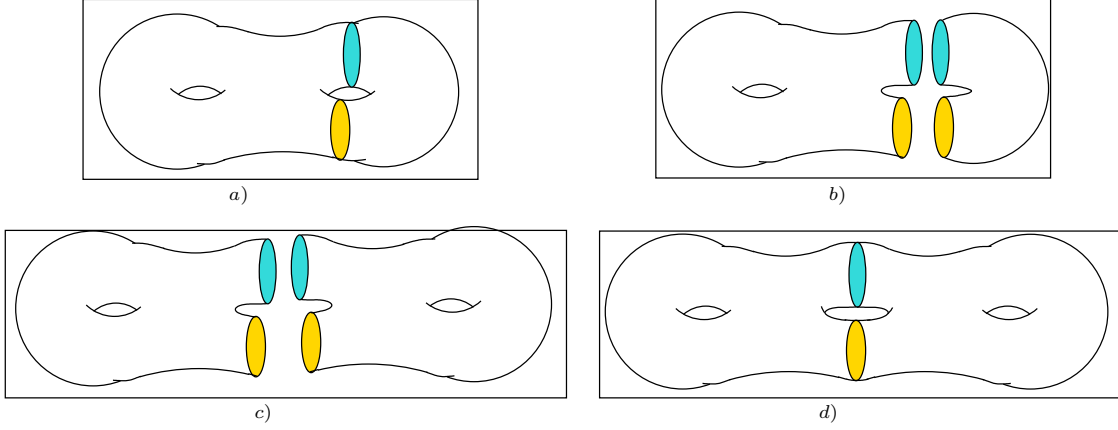


Figure 11: Steps by gluing the new plug.

4.3 Flow through of the faces

We begin by considering the plug 2 described in Figure 2 with the same conditions of subsections 3.1, 3.3.

For this purpose we need to observe with detail the flow behavior through of the faces removed. Indeed, we observe the vector field in the square whose flow is described in Figure 2.

Thus, it will be constructed the new plug through two steps. Firstly, we will be depicted a circle that represents the face 1 on the Cherry flow and let us to study the flow behavior. It should be noted that this vector field exhibits two leaves which belong to the region R and converge to the singularity, i.e., the region R exhibits two singular leaves. Note that these leaves are crossing outward to the face 1. In addition, note that there are trajectories crossing inward to the face 1 too, such as the branch unstable manifold of the singularity. This shows that extensive analysis is necessary for understand the flow behavior to the face 1.

We can observe that the top and bottom region of the singular leaves saturated by the flow are crossing through the face 1, i.e., the flow is pointing outward of the face 1.

By studying the complement of these regions, we have that the behavior of the leaves is depicted as Figure 12. Here, this region exhibits two tangent leaves, whereas the other leaves intersect the region twice, i.e., the other leaves cross and return.

Also, we must research the flow behavior inside to the face 1, but in the complement of Cherry box flow. However, we can to observe that the behavior flow is extended to the whole circle. This finishes the first step.

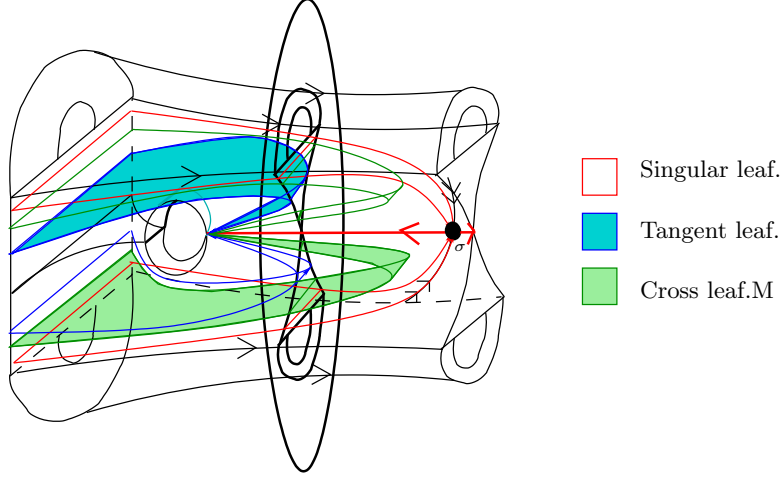


Figure 12: Flow through of the face 1.

We must to observe the flow behavior on the face 2. In this case, is easy to verify that all trajectories are crossing inward to the face 2. Thus, the flow through of the two faces is depicted in the following figure

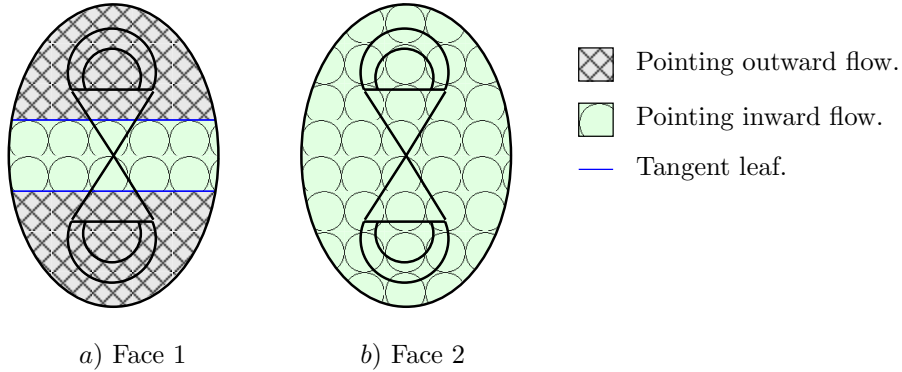


Figure 13: Direction of flow through the faces.

Now, we construct a plug Y containing a singularity σ_2 . Consequently, the dynamical system can be transferred by means of plug Y surgery from one bitorus onto another manifold exporting some of its properties. This singularity generates a hole and this in turns generates a solid tritorus ST_2 in a way that $Y_t(ST_2) \subset \text{Int}(ST_2)$ for all $t > 0$ and Y is transverse to the boundary tritorus. The flow is obtained gluing the plugs 2, 3 with the plug Y as indicated in Figure 14. Indeed, the third hole is generated by the unstable manifold of the singularity σ_2 .

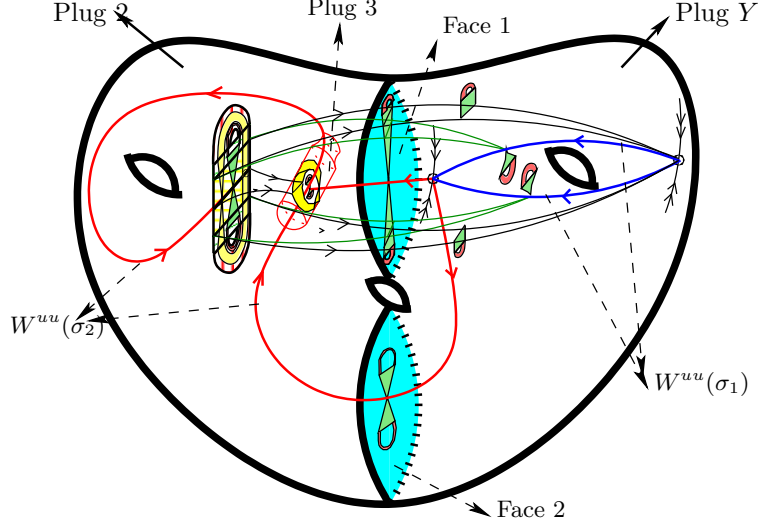


Figure 14: Plug Y

In the same way from the previous subsection, we require some hypotheses for the ambient manifold (after of gluing).

($\hat{X}1$): There are two repelling periodic orbits O_1, O_2 in $Int(ST_2)$ crossing the holes of R .

($\hat{X}2$): There are two solid tori neighborhoods $V_1, V_2 \subset Int(ST_2)$ of O_1, O_2 with boundaries transverse to Y_t such that if $N = ST_2 \setminus (V_1 \cup V_2)$, then N is a compact neighborhood with smooth boundary transverse to Y_t and $Y_t(N) \subset N$ for $t > 0$. As N is a solid tritorus with two solid tori removed, we have that N is connected.

($\hat{X}3$): $R \subset N$ and the return map H induced by Y in R satisfies the properties (G1)-(G3) in Section 3.3. Moreover,

$$\{q \in N : Y_t(q) \notin R, \forall t \in \mathbb{R}\} = Cl(W_Y^{uu}(\sigma_2)).$$

Now, we define

$$\hat{A}^+ = Cl\left(\bigcup_{t \in \mathbb{R}} Y_t(A_G^+)\right) \quad \text{and} \quad \hat{A}^- = Cl\left(\bigcup_{t \in \mathbb{R}} Y_t(A_G^-)\right).$$

By using the Propositions 4.1, 4.3 and Lemma 4.2 we can obtain that the intersection of homoclinic classes is the closure of the unstable manifold of two singularities.

Proof of Theorem B.

By using the Lemma 4.2 and the Proposition 4.3 we have that Y is a sectional Anosov flow and $N(Y)$ is the union of two homoclinic classes $\mathcal{H}_Y^1, \mathcal{H}_Y^2$, where $\mathcal{H}_Y^1 = \hat{A}^+$ and $\mathcal{H}_Y^2 = \hat{A}^-$. It implies that $\mathcal{H}_Y^1 \cap \mathcal{H}_Y^2 = Cl(W_Y^u(\sigma_1) \cup W_Y^u(\sigma_2))$. In particular Y is a Venice mask, and by construction it has two singularities.

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